Sciencia Acta Xaveriana
An International Science Journal
ISSN. 0976-1152

Vol. 1 No. 2
pp. 81–83
Sep. 2010

A Note on a Claim of Rao and Rao for a Pair of Commuting Self-maps

T. Phaneendra and V. Siva Rama Prasad

[1] Visvesvaraya College of Engineering & Technology, RR District–501 510 (AP),
Email: drtp.indra@gmail.com

[2] Nalla Malla Reddy College of Engineering, Uppal, Hyderabad–500 007 (AP)
Email: vangalasrp@yahoo.co.in

Abstract. In obtaining a generalization of a result of Das and Naik (1979) for a pair of commuting self-maps on a complete metric space, I.H. Nagaraja Rao and K.P.R. Rao replaced the continuity condition with a new condition. Further they claimed that the latter was weaker than the continuity. In this note we disprove their argument and give a unified version of the results of Das and Naik and of Rao and Rao.

Keywords: Complete metric space, commuting self-maps, continuity and common fixed point

2000 AMS Subject Classification: 54H25

(Received: 05 May 2010)

As a general result for a pair of commuting self-maps, in 1979 Das and Naik [1] proved the following result:

Theorem 1. *Let S and T be commuting self-maps on a complete metric space X such that*

$$T(X) \subset S(X) \tag{1}$$

satisfying the inequality

$$d(Tx,Ty) \leq \alpha \max\{d(Sx,Sy),d(Sx,Tx),d(Sy,Ty),d(Sx,Ty),d(Sy,Tx)\}$$
$$\text{for all } x,\ y \in X \tag{2}$$

where $0 < \alpha < 1$. *If S is continuous, then S and T will have a unique common fixed point.*

Later, Nagaraja Rao and Rao [2] obtained the conclusion of Theorem 1 by replacing the continuity of S with the condition:

$$d(Tx,Sy) \leq d(y,Sx) \text{ for all } x,y \in X, \text{ where } Sx \neq y. \tag{3}$$

They claimed that the condition (3) is weaker than the continuity of S.

However, we disprove this in the following lines:

Example 1. Let $X, [0,\infty)$ with usual metric d and define $S,T : X \to X$ by $Sx = x/2$ and $Tx = x/3$ for all $x \in X$. Then S and T have a unique common fixed point. In fact, zero is the only common fixed point for them. But for $x = 1$ and $y = 3/8$, we see that $S1 \neq 3/8$ while $d\left(T1, S\frac{3}{8}\right) > d\left(\frac{3}{8}, S1\right)$ showing that the condition (3) fails to hold even if S is continuous. In this case, the common fixed point cannot be obtained by Rao and Rao's result [2, Theorem A]).

Example 2. Consider $X = [0,1]$ with $d = |x-y|$ for all x, y $\in X$ Given $0 < \alpha,\ \beta > 1$ set

$$Sx = \begin{cases} ax, & (0 \leq x < 1) \\ \alpha^2, & (x-1) \end{cases} \quad \text{and} \quad Tx = \begin{cases} \alpha\beta x, & (0 \leq x < 1) \\ \beta\alpha^2, & (x = 1) \end{cases} \quad \text{for all } x \in X.$$

Then (1) and the inequality (2) hold good with $c = \beta$. Also S and T are commuting. It is significant to note that both the self-maps satisfy the condition (3) so that the unique common fixed point, namely zero is obtained by Rao's and Rao's result [2, Theorem A]). But S is not continuous. As such Theorem 1 of Das and Naik [1] is not applicable to find a common fixed point for the self-maps. Thus neither of (3) and the continuity of S is weaker than the other and both of them are independent of each other. Hence the following may be regarded as an appropriate unified version of the Theorem 1 and the result of Rao and Rao: Theorem 1 Suppose S and T are commuting self-maps on a complete metric space X satisfying the inclusion (1) and the inequality (2). If either S is continuous or condition (3) holds, then S and T will have a unique common fixed point.

References

[1] K.M. Das and K.V. Naik, *Common fixed point theorems for commuting maps of a metric space*, Proc. Amer. Math. Soc., **77** (1979) 369–373.

[2] I.H. Nagaraja Rao and K.P.R. Rao, *Unique common fixed point of pair of non linear mappings*, Pure and Appl. Math. Sci., **17** (1983) 63–66.